

Bright solitons in spin-orbit-coupled Bose-Einstein condensates

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(Dated: January 25, 2013)

We study bright solitons in a Bose-Einstein condensate with a spin-orbit coupling that has been realized experimentally. Both stationary bright solitons and moving bright solitons are found. The stationary bright solitons are the ground states and possess well-defined spin-parity, a symmetry involving both spatial and spin degrees of freedom; these solitons are real valued but not positive definite, and the number of their nodes depends on the strength of spin-orbit coupling. For the moving bright solitons, their shapes are found to change with velocity due to the lack of Galilean invariance in the system.

PACS numbers: 03.75.Lm, 03.75.Kk, 03.75.Mn, 71.70.Ej

INTRODUCTION

Solitons are one of the most interesting topics in nonlinear systems. The most fascinating and well-known feature of this localized wave packet is that it can propagate without changing its shape as a result of the balance between nonlinearity and dispersion [1]. The achievement of Bose-Einstein condensation in a dilute atomic gas has offered a clean and parameter-controllable platform to study the properties of solitons [2]. In a Bose-Einstein condensate (BEC), the nonlinearity originates from the atomic interactions and is manifested by the nonlinear term in the Gross-Pitaevskii equation (GPE), which is the mean-field description of BEC [3]. With attractive and repulsive interatomic interactions, the GPE can have bright and dark solitons solutions, respectively. Such dark and bright solitons in BECs have been studied extensively both theoretically [4–10] and experimentally [11–18].

The developments with two-component BECs have further enriched the investigation of solitons in matter waves. The two-component BECs not only introduce more tunable parameters, for example, the interaction between the two species, but also bring in novel nonlinear structures which have no counterparts in the scalar BEC, such as dark-bright solitons (one component is a dark soliton while the other is bright) [19–22], dark-dark solitons [23], bright-bright solitons [24–26], and domain walls [27–29].

Recently, in a landmark experiment, the Spielman group at NIST have engineered a synthetic spin-orbit coupling (SOC) for a BEC [30]. In the experiment, two Raman laser beams are used to couple a two-component BEC. The momentum transfer between lasers and atoms leads to synthetic spin-orbit coupling [31–40]. This kind of spin-orbit coupling has subsequently been realized for neutral atoms in other laboratories [41–44]. These experimental breakthroughs [30, 44, 45] have stimulated extensive theoretical investigation of the properties of spin-

orbit-coupled BECs [46–67], which includes some early studies on solitons. For example, bright-soliton solutions were found analytically for spin-orbit-coupled BECs by neglecting the kinetic energy [68]. Dark solitons for such a system were studied in a one-dimensional ring [69]. In this work we conduct a systematic study of bright solitons for a BEC with attractive interactions and the experimentally realized SOC [30, 42–44]. By solving the GPE both analytically and numerically, we find that these solitons possess a number of novel properties due to the SOC.

In particular, we find that the stationary bright solitons that are the ground state of the system have nodes in their wave function. For a conventional BEC without SOC, its ground state must be nodeless thanks to the “no-node” theorem for the ground state of a bosonic system [70]. Furthermore, these solitons are found to have well-defined spin-parity, a symmetry that involves both spatial and spin degrees of freedom, and can exist in systems with SOC.

We have also found solutions for moving bright solitons. They have the very interesting feature that their shapes change dramatically with increasing velocity. For a conventional BEC, the shape of a soliton does not change with velocity due to the Galilean invariance of the system. In other nonlinear systems, such as Korteweg-de Vries KdV systems, the shape of a soliton changes only in height and width with velocity [71]. In stark contrast, bright solitons in a BEC with SOC can change shape dramatically from nodeless to having many nodes with varying velocity. The new feature arises because of the lack of Galilean invariance due to SOC [62]. It is worthwhile to note that a similar model was proposed a long time ago in the context of nonlinear birefringent fibers [72]. This shows that our results will find applications in nonlinear optics.

MODEL EQUATION

A BEC with the experimentally realized SOC is described by the following GPE

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{1}{2m} (p_x + \hbar \kappa \sigma_y)^2 + \hbar \Delta \sigma_z - g \Psi^\dagger \cdot \Psi \right] \Psi, \quad (1)$$

where the spinor wavefunction $\Psi = (\Psi_1, \Psi_2)^T$, and $\Psi^\dagger \cdot \Psi = |\Psi_1|^2 + |\Psi_2|^2$ with Ψ_1 for up-spin and Ψ_2 for down-spin. The nonlinear coefficient $-g < 0$ is for attractive interatomic interactions, and we have taken $g_{11} = g_{22} = g_{12}$ for simplicity. The SOC is realized experimentally by two counter-propagating Raman lasers that couple two hyperfine ground states Ψ_1 and Ψ_2 . The strength of SOC κ depends on the relative incident angle of the Raman beams and can be changed [65]. The Rabi frequency Δ can be tuned easily by modifying the intensity of the Raman beams. σ are Pauli matrices. A bias homogeneous magnetic field is applied along the y direction. We consider the case that the radial trapping frequency is large and, therefore, the system is effectively one dimensional [14, 15].

For numerical simulation, we rewrite Eq. (1) in a dimensionless form by scaling energy with $\hbar \Delta$ and length with $\sqrt{\hbar/m\Delta}$. The dimensionless GPE is

$$i\partial_t \Phi = \left[-\frac{1}{2} \partial_x^2 + i\alpha \sigma_y \partial_x + \sigma_z - \gamma \Phi^\dagger \cdot \Phi \right] \Phi. \quad (2)$$

The dimensionless parameters $\alpha = -\kappa \sqrt{\hbar \Delta / m}$ and $\gamma = Ng \sqrt{m/(\hbar \Delta)} / \hbar$ with N being the total number of atoms. The dimensionless wavefunctions Φ satisfy $\int dx (|\Phi_1|^2 + |\Phi_2|^2) = 1$. The SOC term $i\alpha \sigma_y \partial_x$ in Eq. (2) indicates that spin σ_y only couples the momentum in the x direction. The energy functional of our system is

$$E = \int dx \left[\frac{1}{2} |\partial_x \Phi_1|^2 + \frac{1}{2} |\partial_x \Phi_2|^2 + |\Phi_1|^2 - |\Phi_2|^2 + \alpha \Phi_1^* \partial_x \Phi_2 - \alpha \Phi_2^* \partial_x \Phi_1 - \frac{\gamma}{2} (|\Phi_1|^4 + |\Phi_2|^4 + 2|\Phi_1|^2 |\Phi_2|^2) \right]. \quad (3)$$

STATIONARY BRIGHT SOLITONS

We focus on the simplest stationary bright solitons, which are the ground states of the system. To find these solitons, we solve Eq. (2) by using an imaginary time-evolution method. Two typical bright solitons are shown in Fig. 1. One interesting feature is immediately noticed. There are “nodes” in these ground-state bright solitons. This is very different from the conventional BEC, where there are no nodes in this kind of ground state soliton as demanded by the no-node theorem for the ground state of a boson system. Our results confirm that this no-node theorem does not hold for systems with SOC [70].

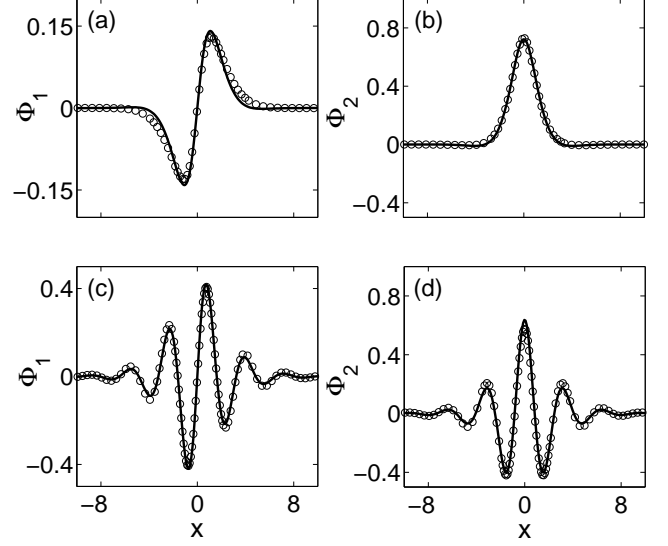


FIG. 1: Stationary bright solitons at $\gamma = 1.0$. The solid lines are numerical results and the circles are from the variational method. In (a) and (b), $\alpha = 1.0$. In (c) and (d), $\alpha = 2.0$.

There can exist a unique symmetry for systems with SOC, spin-parity, which involves both spatial and spin degrees of freedom. The operator for spin-parity is defined as

$$\mathcal{P} = P \sigma_z, \quad (4)$$

where P is the parity operator. It is easy to verify that our system is invariant under the action of spin-parity \mathcal{P} . By direct observation, one can see that the bright solitons shown in Fig. 1 satisfy

$$\mathcal{P} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = - \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}, \quad (5)$$

as the up component Φ_1 has odd parity while the other component Φ_2 is even. Therefore, these bright solitons have spin-parity -1 . In fact, all the ground state bright solitons that we have found have spin-parity -1 . That the eigenvalue of \mathcal{P} for these solitons is -1 and not 1 can be understood in the following manner. When the strength of SOC α decreases to zero, the up component Ψ_1 shrinks to zero and only the down component survives. Since the system becomes a conventional BEC without SOC, the no-node theorem demands that the surviving down component has even symmetry. As the SOC is turned up continuously and slowly, the symmetry of the second component should remain and the spin parity has to be -1 .

For a more detailed analysis of these bright solitons, we attempt to find an analytical approximation for the wavefunctions using the variational method. Motivated by the features of the stationary bright solitons shown in

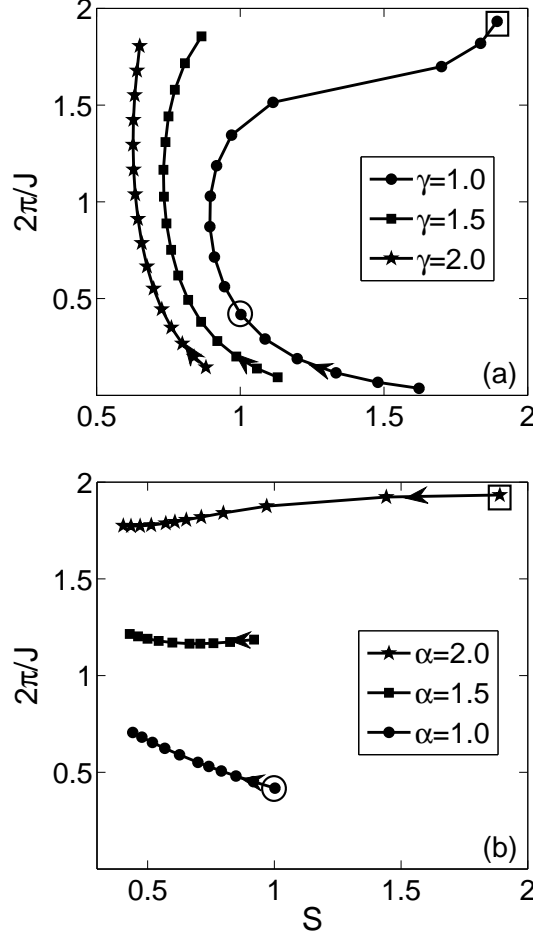


FIG. 2: The relation between the number of soliton nodes $2\pi/J$ and the soliton width S . (a) Circles, squares, and stars are for $\gamma = 1.0, 1.5, 2.0$, respectively. Here α increases from 0.5 to 2.0 (the arrow direction). (b) Circles, squares, and stars are for $\alpha = 1.0, 1.5, 2.0$. Here γ increases from 1.0 to 4.0 (the arrow direction). The open circle and square correspond to the bright solitons in Figs. 1(a), and 1(b) and Figs. 1(c), and 1(d), respectively.

Fig. 1, we propose the following trial wave functions for these solitons:

$$\Phi = \left(\frac{A \sin(2\pi x/J)}{B \cos(2\pi x/J)} \right) \text{sech}(x/S). \quad (6)$$

The parameters A , B , J , and S are determined by minimizing the energy functional in Eq. (3) with the constraining normalization. The results of the trial wave functions are compared with the numerical results in Fig. 1, where it can be seen that they are in excellent agreement.

It is clear from Eq. (6) that the parameter $2\pi/J$ can be regarded roughly as the number of nodes in the bright

solitons, while S is for the overall width of the soliton. Both of them depend on the SOC strength α and the interaction strength γ . In Fig. 2 we have plotted the relation between $2\pi/J$ and S , demonstrating how the number of nodes is related to the soliton width for different values of α and γ . As shown in the Fig. 2, for solitons with the same number of nodes, they are wider for smaller interaction strength γ [Fig. 2(a)]; the solitons with the same width have more nodes for larger SOC strength α [Fig. 2(b)].

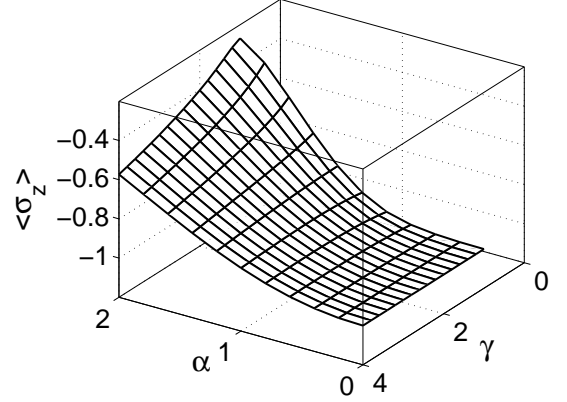


FIG. 3: The spin polarization $\langle \sigma_z \rangle$ as a function of α and γ .

It has been reported that there exists a quantum phase transition for the ground states in the spin-orbit-coupled system with repulsive interaction [65]. It is interesting to check whether such a phase transition exists for the case of attractive interaction. For this purpose, we have computed the spin polarization $\langle \sigma_z \rangle = \int dx (\Phi_1^2 - \Phi_2^2)$ for these bright solitons and the results are plotted in Fig. 3. We see that for a given γ , it changes smoothly with the SOC strength α . For the cases of repulsive interaction and no interaction, the spin polarization is found to change sharply with α , indicating a quantum phase transition [65]. The smooth behavior of Fig. 3 suggests there is no quantum phase transition.

MOVING BRIGHT SOLITONS

After the study of stationary bright solitons, we turn our attention to moving bright solitons. For a conventional BEC without SOC, it is straightforward to find a moving bright soliton from a stationary soliton: if the wave function Φ_s describes a stationary soliton, then $\exp(ivx)\Phi_s(x - vt)$ is the wave function, up to a trivial phase, for a soliton moving at speed v . This is due to the invariance of the system under Galilean transformations.

However, Galilean invariance is violated for a spin-orbit-coupled BEC [62]. To see this explicitly, we assume

moving solitons having the following form:

$$\Phi_M(x, t) = \Phi_v(x - vt, t) \exp(ivx - i\frac{1}{2}v^2t), \quad (7)$$

where Φ_v is a localized function. Substitution of $\Phi_M(x, t)$ into Eq. (2) yields

$$i\partial_t \Phi_v = \left[-\frac{1}{2}\partial_x^2 + \alpha\sigma_y(i\partial_x - v) + \sigma_z - \gamma\Phi_v^\dagger \cdot \Phi_v \right] \Phi_v. \quad (8)$$

Compared to Eq. (2), this dynamical equation has an additional term $\alpha v\sigma_y$, indicating the violation of Galilean invariance. This violation means that it is no longer a trivial task to find a moving bright soliton for a BEC with SOC.

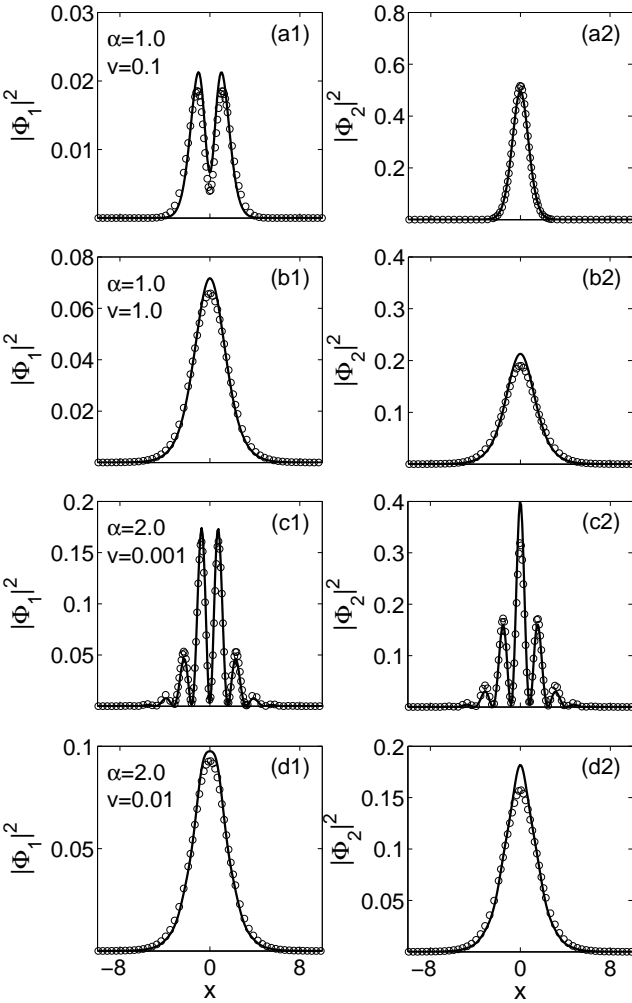


FIG. 4: Moving bright soliton profiles $\Phi_v(x)$ from the numerical calculation (solid line) and the variational method (circles) with Eq. (9). $\gamma = 1.0$. In (a) and (b), $\alpha = 1.0$. In (c) and (d) $\alpha = 2.0$.

To find moving bright solitons, we numerically solve Eq. (8) using the imaginary time-evolution method. Two

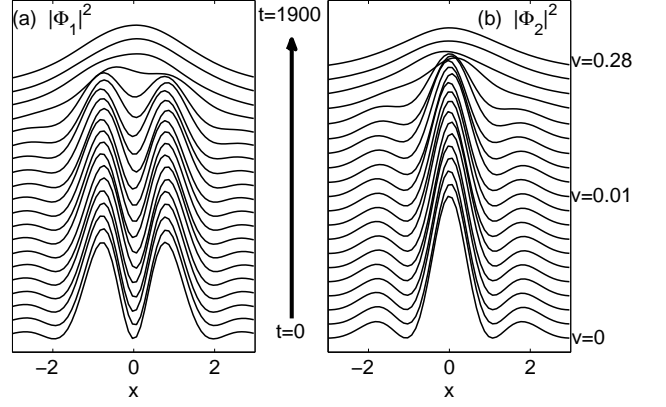


FIG. 5: Dynamical evolution of a bright soliton under the influence of a small linear potential $V(x) = \eta x$. $\alpha = 1.7$, $\gamma = 1.0$, $\eta = 0.001$. For better comparison of shapes, the centers of the soliton densities have all been shifted to the zero point. The velocity of the soliton is labeled at the right of (b).

typical moving bright solitons are shown in Fig. 4, where we see clearly that the shapes of moving bright solitons change with their velocities. As seen in Figs. 4(a) and 4(b), when the velocity v is changed from 0.1 to 1, the density of the up component changes from having two peaks to having only one. At a larger SOC strength, such as $\alpha = 2$ in Figs. 4(c) and 4(d), a small change in velocity leads to a dramatic change in the soliton profiles.

Similar to the stationary soliton, these moving bright solitons can also be found with the variational method by minimizing the energy functional with the following trial wave function:

$$\Phi_v(x) = \left(\frac{A \left[\sin \frac{2\pi x}{J} + \rho_1 i \cos \frac{2\pi x}{J} \right]}{B \left[\cos \frac{2\pi x}{J} + \rho_2 i \sin \frac{2\pi x}{J} \right]} \right) \text{sech} \frac{x}{S}, \quad (9)$$

with two new parameters ρ_1 and ρ_2 . When $\rho_1 = \rho_2 = 0$, we recover the stationary soliton in Eq. (6). The solutions obtained with the variational method are plotted in Fig. 4 and they agree well with the numerical results. It is clear from the trial wave function that the moving bright soliton has no well-defined spin-parity \mathcal{P} .

These moving bright solitons are adiabatically linked to the stationary bright solitons. To see this, we slowly accelerate the stationary bright soliton by adding a small linear potential in Eq. (2) integrating with a stationary bright soliton as the initial condition. The dynamical evolution of this soliton is shown in Fig. 5, where we see clearly how a stationary soliton is developed into a moving soliton with its shape changing constantly. Note that the centers of solitons in Fig. 5 have all been shifted to the zero point for better comparison between shapes.

We note that there are other solitons, which also change their shapes with velocity, for example, dark solitons in a BEC and bright solitons in the KdV system [71]. This change is also caused by the lack of the Galilean

invariance in the system. For the dark soliton, the constant background provides a preferred reference frame and breaks the Galilean invariance. In the KdV system, the violation is caused by the non-quadratic linear dispersion. However, in these systems, the change in shape with velocity is not as dramatic: there are only changes in the height and width of the solitons. With a spin-orbit-coupled BEC, the number of peaks in the solitons can change with a slight change in velocity.

CONCLUSION

We have systematically studied both stationary and moving bright solitons in a spin-orbit-coupled BEC. These bright solitons have features not present without spin-orbit coupling, for example, the existence of nodes and spin-parity in the stationary bright solitons, and the change in shape with velocity in the moving bright solitons. Although there are multiple peaks in the soliton profiles, the bright solitons that we have found are single solitons. It would be very interesting to seek out multiple-solitons solutions for this spin-orbit-coupled system. These bright solitons should be able to be observed in experiment. One can apply the same Raman laser setup to generate the synthetic spin-orbit coupling for an optical dipole-trapped ^7Li condensate where the interatomic interaction is attractive by nature.

ACKNOWLEDGMENTS

We would like to thank Matthew Davis for critical readings and insightful comments. Y.X. thanks Y. Hu and M. Gong for very helpful discussions. Y.X. and B.W. are supported by the NBRP of China (Grants No. 2012CB921300 and No. 2013CB921900), the NSF of China (Grants. No. 10825417 and No. 11274024), the RFDP of China (Grant No. 20110001110091).

Note added. Recently, we noticed a newly posted preprint on bright solitons in a BEC with SOC [73]. Some of their results overlap with ours.

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